converged. The analysis terminates when all components are converged in this sense. Of course, the entire error vector must be examined on each convergence test.

This scheme naturally accommodates both large and small current components. Large components usually converge on the basis of the relative error, and small ones on the fractional error, as the relative criterion is a weaker one for large components and the absolute criterion is weaker for small components. It is a simple matter to select the criteria so they ensure that all errors are sufficiently small at termination.

3.3.9.8 Initial Estimate

One important property of Newton's method is that its speed and reliability of convergence depend strongly upon the initial estimate of the solution vector. Formulating the initial estimate may not be difficult in analyzing a specific type of circuit, but it may be difficult to conceive of a way to form initial estimates in a general-purpose circuit-analysis program, which must accommodate a wide variety of circuits that have a concomitant variety of possible responses.

For nearly linear circuits, such as class-A power amplifiers, the linear response is a good initial estimate. The response can be found by setting the excitation level to a small value and the harmonic number, K, equal to one, so the size of the problem is relatively small. When the solution has completed, the results are scaled to the correct excitation level and K is reset to the desired value for the large-signal analysis.

In strongly nonlinear circuits, such as class-B or -C amplifiers, frequency multipliers, and mixers, an initial estimate is more difficult to generate. Occasionally the nature of the circuit allows a good estimate; for example, in diode mixers, the diode-voltage waveform invariably is a clipped sinusoid. In difficult cases, it may be best first to do a dc analysis, then to apply the RF signal and increase it using a continuation method.

3.4 LARGE-SIGNAL/SMALL-SIGNAL ANALYSIS USING CONVERSION MATRICES

Large-signal/small-signal analysis, or *conversion matrix analysis*, is useful for a large class of problems wherein a nonlinear device is driven, or "pumped," by a single large sinusoidal signal; another signal, much smaller, is applied; and we seek only the linear response to the small signal. The most common application of this technique is in the design of mixers and in nonlinear noise analysis. The process involves first analyzing the nonlinear device under large-signal excitation only, usually by the harmonic-balance method. The nonlinear elements in the device's equivalent circuit are then linearized to create small-signal, linear, time-varying elements, and finally a small-signal analysis is performed. The method is much more efficient than multitone harmonic-balance analysis but provides only the linear response of the circuit. It cannot be used for determining saturation or intermodulation distortion in mixers, but it is a good method for calculating a mixer's conversion efficiency and its RF and IF port impedances. The results of the harmonic-balance analysis can be used for finding LO voltage and current waveforms, and LO port impedance.

3.4.1 Conversion Matrix Formulation

Figure 3.9 shows a nonlinear resistive element driven by a large-signal voltage, V, generating a current I. The nonlinear element has the I/V relationship $I = f(V)$. Following the process outlined in Chapter 2, we can find the incremental small-signal current by assuming that V consists of the sum of a large-signal component V_0 and a small-signal component v. The current resulting from this excitation can be found by expanding $f(V_0 + v)$ in a Taylor series,

$$\begin{aligned} f(V_0 + v) &= f(V_0) + \frac{d}{dV} f(V) \bigg|_{V = V_0} v + \frac{1}{2} \frac{d^2}{dV^2} f(V) \bigg|_{V = V_0} v^2 \\ &\quad + \frac{1}{6} \frac{d^3}{dV^3} f(V) \bigg|_{V = V_0} v^3 + \ldots \end{aligned} \tag{3.94}$$

The small-signal, incremental current is found by subtracting the large-signal component of the current,

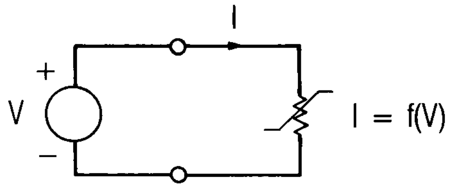

Figure 3.9 Nonlinear resistive element driven by a large excitation.

$$i(v) = I(V_0 + v) - I(V_0) \tag{3.95}$$

If $v << V_0$, v^2, v^3, ... are negligible (and, in any event, are nonlinear, so they do not contribute to the linear response). Then,

$$i(v) = \frac{d}{dV} f(V)\bigg|_{V = V_0} v \tag{3.96}$$

V_0 need not be a dc quantity; it can be a time-varying large-signal voltage $V_L(t)$ (in fact, V_0 and V_L are control voltages). We assume that this is the case, and also that $v = v(t)$, a function of time. Then

$$i(t) = \frac{d}{dV} f(V)\bigg|_{V = V_L(t)} v(t) \tag{3.97}$$

Equation (3.97) can be expressed as

$$i(t) = g(t)v(t) \tag{3.98}$$

The time-varying conductance in (3.98), $g(t)$, is the derivative of the element's I/V characteristic at the large-signal voltage. This is the usual definition of small-signal conductance for static elements. By an analogous derivation, one could have a current-controlled resistor with the V/I characteristic

$$V = f_R(I) \tag{3.99}$$

and obtain the small-signal v/i relation

$$v(t) = r(t)i(t) \tag{3.100}$$

where

$$r(t) = \frac{d}{dI} f_R(I)\bigg|_{I = I_L(t)} \tag{3.101}$$

Often, the nonlinear element is a function of more than one control voltage. A conductance controlled by two voltages has $I = f_2(V_1, V_2)$. $f_2(V_1, V_2)$ can be expanded in a two-dimensional Taylor series, and after subtracting the large-signal current component and retaining only the linear terms,

$$i(t) = g_1(t)v_1(t) + g_2(t)v_2(t) \tag{3.102}$$

where

$$\begin{aligned} g_1(t) &= \left.\frac{\partial}{\partial V_1} f_2(V_1, V_2)\right|_{\substack{V_1 = V_{L,1}(t) \\ V_2 = V_{L,2}(t)}} \\ g_2(t) &= \left.\frac{\partial}{\partial V_2} f_2(V_1, V_2)\right|_{\substack{V_1 = V_{L,1}(t) \\ V_2 = V_{L,2}(t)}} \end{aligned} \tag{3.103}$$

Equation (3.102) shows that a nonlinear conductance having two control voltages is equivalent to two conductances in parallel. One must be a controlled current source, and the other may be either a controlled source or a time-varying two-terminal conductance. When the I/V characteristic is a function of more than two voltages, (3.102) can be extended in the manner one would expect:

$$i(t) = g_1(t)v_1(t) + g_2(t)v_2(t) + g_3(t)v_3(t) + \ldots \tag{3.104}$$

It is unusual, however, to encounter a nonlinear element having more than two control voltages.

The same process can be followed with a capacitor. A nonlinear capacitor has the Q/V characteristic $Q = f_Q(V)$, and by a similar derivation, the incremental, small-signal charge is

$$q(t) = \left.\frac{d}{dV} f_Q(V)\right|_{V = V_L(t)} v(t) \tag{3.105}$$

or

$$q(t) = c(t)v(t) \tag{3.106}$$

The capacitor's current is the time derivative of the charge:

$$i(t) = \frac{d}{dt}q(t) = c(t)\frac{d}{dt}v(t) + v(t)\frac{d}{dt}c(t) \tag{3.107}$$

Like a conductance, a capacitance can have multiple control voltages. In a manner analogous to (3.102) to (3.104), the small-signal charge is

$$q(t) = c_1(t)v_1(t) + c_2(t)v_2(t) + c_3(t)v_3(t) + \ldots \tag{3.108}$$

and the current is found by differentiating with respect to time:

$$\begin{aligned} i(t) = \frac{d}{dt}q(t) &= c_1(t)\frac{d}{dt}v_1(t) + v_1(t)\frac{d}{dt}c_1(t) \\ &+ c_2(t)\frac{d}{dt}v_2(t) + v_2(t)\frac{d}{dt}c_2(t) + \ldots \end{aligned} \tag{3.109}$$

A nonlinear element excited by two tones supports currents and voltages at the mixing frequencies $m\,\omega_1 + n\,\omega_2$, where m and n are integers. If we assume that one of those tones, ω_1, has such a low level that it does not generate harmonics, and the other is a large-signal sinusoid at ω_p, the mixing frequencies are $\omega = \pm\omega_1 + n\,\omega_p$. This equation represents the set of frequency components shown in Figure 3.10, which consists of two tones on either side of each large-signal harmonic frequency, separated by $\omega_0 = |\omega_1 - \omega_p|$. A more compact representation of the mixing frequencies is

$$\omega_n = \omega_0 + n\omega_p \tag{3.110}$$

which is shown in Figure 3.11 and includes only half of the mixing frequencies: the negative components of the lower sidebands and the positive components of the upper sidebands. This set of frequencies is adequate for two reasons: first, the small-signal analysis is linear, so by the superposition principle, the results for positive and negative components can be separated; and second, positive- and negative-frequency components are complex conjugate pairs, so knowledge of only one is

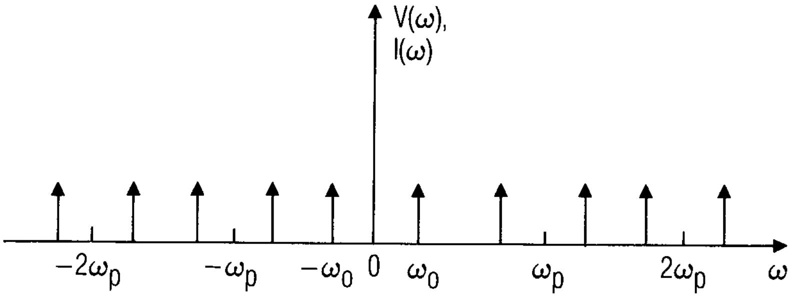

Figure 3.10 Spectrum of small-signal mixing frequencies in the pumped nonlinear element.

necessary. We will carry only the components in (3.110) in the following analysis, with confidence that the others can be generated when necessary.

The frequency-domain currents and voltages in a time-varying circuit element are related by a *conversion matrix*. We begin by deriving the conversion matrix that represents a time-varying conductance. The small-signal voltage and current can be expressed in the frequency notation of (3.110) as

$$v'(t) = \sum_{n=-\infty}^{\infty} V_n \exp(j\omega_n t) \tag{3.111}$$

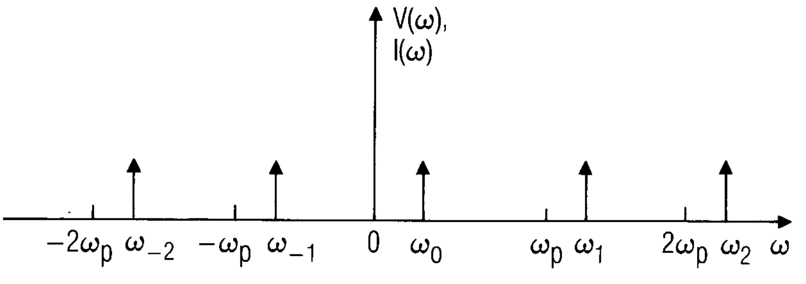

Figure 3.11 Spectrum of small-signal mixing frequencies illustrating the frequency notation of (3.110).

and

$$i'(t) = \sum_{n=-\infty}^{\infty} I_n \exp(j\omega_n t) \tag{3.112}$$

where the primes indicate that $v'(t)$ and $i'(t)$ are sums of the positive- and negative-frequency phasor components in (3.110) and are not the complete time waveforms. Above all, (3.111) and (3.112) are not Fourier series, in spite of their superficial resemblance. The conductance waveform $g(t)$ can be expressed by its Fourier series,

$$g(t) = \sum_{n=-\infty}^{\infty} G_n \exp(jn\omega_p t) \tag{3.113}$$

and the voltage and current are related by Ohm's law,

$$i'(t) = g(t)v'(t) \tag{3.114}$$

Substituting (3.111) through (3.113) into (3.114) gives the relation,

$$\sum_{k=-\infty}^{\infty} I_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_n V_m \exp(j\omega_{m+n} t) \tag{3.115}$$

Equating terms on both sides of the equation in (3.115) results in a set of equations that can be expressed in matrix form:

$$\begin{bmatrix} I^*_{-N} \\ I^*_{-N+1} \\ I^*_{-N+2} \\ \cdots \\ \cdots \\ I^*_{-1} \\ I_0 \\ I_1 \\ \cdots \\ \cdots \\ I_N \end{bmatrix} = \begin{bmatrix} G_0 & G_{-1} & G_{-2} & \cdots & G_{-2N} \\ G_1 & G_0 & G_{-1} & \cdots & G_{-2N+1} \\ G_2 & G_1 & G_0 & \cdots & G_{-2N+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ G_{N-1} & G_{N-2} & G_{N-3} & \cdots & G_{-N-1} \\ G_N & G_{N-1} & G_{N-2} & \cdots & G_{-N} \\ G_{N+1} & G_N & G_{N-1} & \cdots & G_{-N+1} \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ G_{2N} & G_{2N-1} & G_{2N-2} & \cdots & G_0 \end{bmatrix} \begin{bmatrix} V^*_{-N} \\ V^*_{-N+1} \\ V^*_{-N+2} \\ \cdots \\ \cdots \\ V^*_{-1} \\ V_0 \\ V_1 \\ \cdots \\ \cdots \\ V_N \end{bmatrix} \tag{3.116}$$

Two details in (3.116) must be clarified. First, the vectors in (3.116) have been truncated to a limit of $n = N$ for I_n and V_n, and $n = 2N$ for G_n. We assume that V_n, I_n, and G_n are negligible beyond these limits. The second detail is that the negative-frequency components (V_n, I_n where $n < 0$) are shown as conjugate. The conjugates are caused by a change of definition; according to (3.110), ω_n is negative when $n < 0$, so the I_n and V_n are negative-frequency components when $n < 0$. We would rather define them as phasors, which are always positive-frequency components. Positive- and negative-frequency components are related as $V_{-n} = V_n^*$ and $I_{-n} = I_n^*$, so if we wish V_n, I_n to represent positive-frequency components, they must be V_n^*, I_n^*. Thus the conversion matrix relates ordinary phasor voltages to currents at each mixing frequency. The main advantage of making this change is that the conversion matrix is now completely compatible with conventional linear, sinusoidal steady-state analysis.

The dual case, a time-varying resistor, has an unsurprising result. The conversion matrix is

$$\begin{bmatrix} V^*_{-N} \\ V^*_{-N+1} \\ V^*_{-N+2} \\ \cdots \\ \cdots \\ V^*_{-1} \\ V_0 \\ V_1 \\ \cdots \\ \cdots \\ V_N \end{bmatrix} = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots & R_{-2N} \\ R_1 & R_0 & R_{-1} & \cdots & R_{-2N+1} \\ R_2 & R_1 & R_0 & \cdots & R_{-2N+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{N-1} & R_{N-2} & R_{N-3} & \cdots & R_{-N-1} \\ R_N & R_{N-1} & R_{N-2} & \cdots & R_{-N} \\ R_{N+1} & R_N & R_{N-1} & \cdots & R_{-N+1} \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ R_{2N} & R_{2N-1} & R_{2N-2} & \cdots & R_0 \end{bmatrix} \begin{bmatrix} I^*_{-N} \\ I^*_{-N+1} \\ I^*_{-N+2} \\ \cdots \\ \cdots \\ I^*_{-1} \\ I_0 \\ I_1 \\ \cdots \\ \cdots \\ I_N \end{bmatrix} \tag{3.117}$$

where the R_n are the Fourier components of the resistance waveform. As one might expect, the resistance-form conversion matrix of any element is the inverse of its conductance-form matrix, as long as the element can be defined either as a time-varying conductance or resistance.

The conversion matrix of a capacitor is only slightly more complicated. The capacitor’s charge is given by

$$q'(t) = c(t)v'(t) \tag{3.118}$$

and $c(t)$ has the Fourier series

$$c(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_p t) \tag{3.119}$$

The current is

$$i'(t) = \frac{d}{dt}q'(t) \tag{3.120}$$

and $q'(t)$ has the form

$$q'(t) = \sum_{n=-\infty}^{\infty} Q_n \exp(j\omega_n t) \tag{3.121}$$

Substituting (3.111), (3.119), and (3.121) into (3.118) gives

$$\sum_{k=-\infty}^{\infty} Q_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_n V_m \exp(j\omega_{m+n} t) \tag{3.122}$$

The current can be found by differentiating. In the frequency domain, differentiation corresponds to multiplying by $j\omega$, so

$$\sum_{k=-\infty}^{\infty} I_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} j\omega_{m+n} C_n V_m \exp(j\omega_{m+n} t) \tag{3.123}$$

Equating terms at the same frequency gives the matrix equation

$$\mathbf{I} = j\Omega\mathbf{C}\mathbf{V} \tag{3.124}$$

where **I** and **V** represent the frequency-component current and voltage vectors and C represents the conversion matrix for the capacitance. **I** and **V** are identical to the vectors in (3.116) and (3.117), and **C** has the same form as the conductance and resistance matrices in those equations. The matrix Ω is a diagonal matrix; its elements are $j\omega_{-N}$ to $j\omega_N$:

$$\Omega = \begin{bmatrix} j\omega_{-N} & 0 & \dots & 0 \\ 0 & j\omega_{-N+1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & j\omega_N \end{bmatrix} \tag{3.125}$$

3.4.1.1 Example: Conversion Matrix of a Time-Varying Element

We form the conversion matrix of the circuit shown in Figure 3.12(a). It consists of a conductance in series with a switch; the switch is opened and closed with a duty cycle of 0.5, so the combination has the waveform shown in Figure 3.12(b). Its Fourier series, when $t_0 = 0.5T$, is

$$\begin{aligned} g(t) = G_p[&0.5 + 0.318\exp(j\omega_p t) + 0.318\exp(-j\omega_p t) \\ &- 0.106\exp(j3\omega_p t) - 0.106\exp(-j3\omega_p t) \\ &+ 0.064\exp(j5\omega_p t) + 0.064\exp(-j5\omega_p t) + \ldots \end{aligned} \tag{3.126}$$

The conversion matrix when $2N = 6$ is

$$\mathbf{G} = G_p \begin{bmatrix} 0.5 & 0.318 & 0 & -0.106 & 0 & 0.064 & 0 \\ 0.318 & 0.5 & 0.318 & 0 & -0.106 & 0 & 0.064 \\ 0 & 0.318 & 0.5 & 0.318 & 0 & -0.106 & 0 \\ -0.106 & 0 & 0.318 & 0.5 & 0.318 & 0 & -0.106 \\ 0 & -0.106 & 0 & 0.318 & 0.5 & 0.318 & 0 \\ 0.064 & 0 & -0.106 & 0 & 0.318 & 0.5 & 0.318 \\ 0 & 0.064 & 0 & -0.106 & 0 & 0.318 & 0.5 \end{bmatrix} \tag{3.127}$$

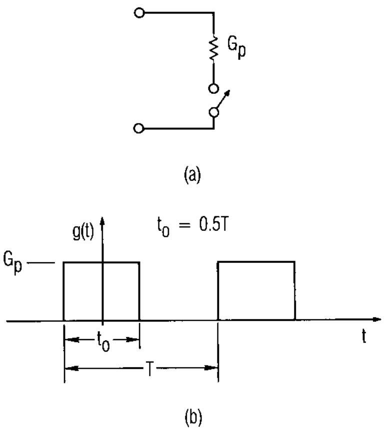

Figure 3.12 (a) Time-varying conductance; (b) conductance waveform, $g(t)$.

which relates the mixing products up to ω_3, those close to the third harmonic of the large-signal excitation.

3.4.2 Applying Conversion Matrices to Time-Varying Circuits

In order to mix ordinary, constant-value, and time-varying components in the same equations, the constant-value elements must have a conversion matrix form. This form is a diagonal matrix, and the element value must occupy all the locations on the main diagonal. The conversion matrix of a frequency-sensitive time-invariant element, such as a fixed impedance or admittance, is also a diagonal; however, the matrix elements are the impedance or admittance at the frequency corresponding to the location in the matrix. For example, the impedance-form conversion matrix of a static, lumped impedance is

$$\mathbf{Z} = \begin{bmatrix} Z^*(-\omega_{-N}) & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & Z^*(-\omega_{-N+1}) & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Z^*(-\omega_{-1}) & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & Z^*(\omega_0) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & Z(-\omega_1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & Z(\omega_N) \end{bmatrix} \tag{3.128}$$

When $n < 0$ ω_n is negative, so the impedance or admittance in the ω_n position is $V_n^*/I_n^* = Z^*(-\omega_n)$; thus the entry must be the conjugate of the positive-frequency impedance or admittance at that frequency.

Equations (3.116) and (3.117) can be expressed, like (3.124), as

$$\mathbf{I} = \mathbf{G}\mathbf{V} \tag{3.129}$$

$$\mathbf{V} = \mathbf{R}\mathbf{I} \tag{3.130}$$

These relations have the same form as those that define the *I*/*V* relations of linear, time-invariant resistance, conductance, and capacitance in the sinusoidal steady state. The only difference is that these are matrix equations, and the latter are scalar. The individual current and voltage components in the **V** and **I** vectors must satisfy Kirchoff's current and voltage laws in any linear circuit using time-varying elements, just as in time-invariant circuits. Therefore, the matrix equations can be used in exactly the same way as the scalar ones, as long as the requirements of matrix arithmetic are met: the order of multiplication must be preserved, and one must invert and multiply instead of dividing.

This realization allows all the tools of conventional sinusoidal, steady-state analysis to be applied to time-varying circuits. For example, the conversion matrix for two elements in parallel is the sum of their individual admittance-form matrices, and for two elements in series, it is the sum of their impedance-form matrices. One can also generate transfer functions and input/output impedances or admittances in terms of conversion matrices.

A second property of the conversion matrices is that they can be treated in all ways like multiport admittance or impedance matrices; the "ports" in the conversion matrix are currents and voltages at different frequencies, not physically separate ports. In theory, one could separate the frequency components by filters and create a physically separate port for each, without changing any of the circuit's properties. Indeed, in designing components that include time-varying elements, such as mixers, one tries to separate at least a few of the frequency components in this manner, in order to realize input and output ports, and to terminate other mixing products optimally. This property allows multiport-circuit concepts to be employed in interconnecting time-varying circuits, interfacing them with matching networks, and determining their gain, impedances, and stability. One can even convert the admittance- or impedance-form conversion matrix to an S-parameter form. These points are illustrated by the following examples.

3.4.2.1 Example: Conversion Matrix of a Simple Circuit

We derive the conversion matrix that represents the circuit shown in Figure 3.13. This circuit consists of a time-varying conductance and capacitance in parallel and a resistor in series (this is a common model of a pumped mixer diode). We assume that a large-signal analysis has been performed, and that the time waveforms and conversion matrices of each circuit element have been determined. $\mathbf{C}_j$ and $\mathbf{G}_j$ are the conversion matrices representing $c_j(t)$ and $g_j(t)$, respectively.

Because the capacitor and conductance are in parallel, the conversion matrix is the sum of the admittance-form conversion matrices of each component:

$$\mathbf{Y}_j = \mathbf{G}_j + j\Omega\mathbf{C}_j \tag{3.131}$$

and their impedance-form conversion matrix is the inverse:

$$\mathbf{Z}_j = \mathbf{Y}_j^{-1} = (\mathbf{G}_j + j\Omega\mathbf{C}_j)^{-1} \tag{3.132}$$

The conversion matrix for the resistor is $R\mathbf{1}$, where $\mathbf{1}$ is the $2N + 1 \times N + 1$ identity matrix. $R\mathbf{1}$ is in series with $\mathbf{Z}_j$, so the impedance-form conversion matrix of the entire circuit is the sum of $R\mathbf{1}$ and $\mathbf{Z}_j$:

$$\mathbf{Z}_c = R\mathbf{1} + \mathbf{Z}_j = R\mathbf{1} + (\mathbf{G}_j + j\Omega\mathbf{C}_j)^{-1} \tag{3.133}$$

The admittance-form matrix, if needed, is just the inverse of the impedance-form matrix.

3.4.2.2 Example: Two-Port Conversion Matrix

We calculate the conversion matrix that represents the simplified FET equivalent circuit shown in Figure 3.14(a), which could represent a FET mixer. It has two nonlinear circuit elements, $I_d(V_g, V_d)$ and $C_g(V_g)$, and all the remaining elements are linear. The circuit is treated as a two-port, so a two-port admittance-form matrix is needed. It has the form

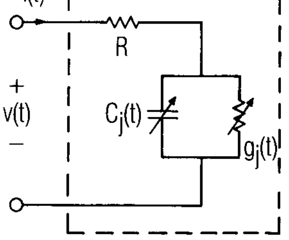

Figure 3.13 Pumped diode equivalent circuit of the example.

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{1,1} & \mathbf{Y}_{1,2} \\ \mathbf{Y}_{2,1} & \mathbf{Y}_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \tag{3.134}$$

where $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{V}_1$, and $\mathbf{V}_2$ are current and voltage vectors as shown in (3.116) and (3.117), and the $\mathbf{Y}_{m,n}$ submatrices are each complete conversion matrices. Thus, (3.134) relates not only the currents and voltages at the mixing frequencies and at each port, but also includes transfer terms between ports.

Again, we assume that a large-signal analysis has been performed and that the nonlinear elements have been converted to their incremental, time-varying forms. The drain current source can be split into two elements, $g_m(t)$ and $g_d(t)$, according to (3.102) and (3.103); the former element is a controlled source, representing the time-varying transconductance, and the latter is a time-varying drain-to-source conductance. The resulting circuit is shown in Figure 3.14(b).

The submatrices are defined in a manner entirely analogous to static admittance matrices:

$$\mathbf{I}_1 = \mathbf{Y}_{1,1}\mathbf{V}_1 \qquad \mathbf{V}_2 = \mathbf{0} \tag{3.135}$$

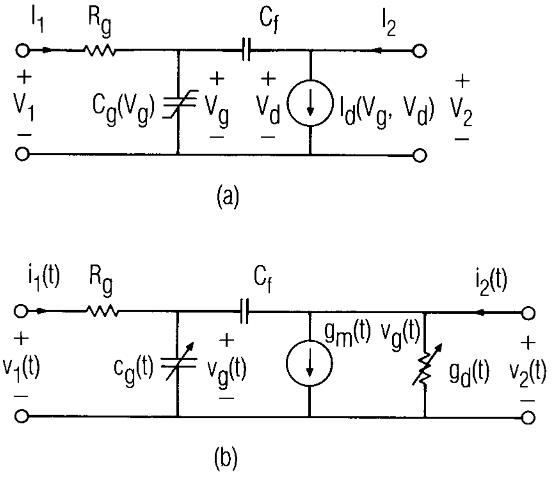

Figure 3.14 (a) FET nonlinear equivalent circuit for the example; (b) time-varying linear equivalent circuit.

and so on, where $\mathbf{0}$ is the zero vector. The time-varying quantities $c_g(t)$, $g_m(t)$, and $g_d(t)$ have conversion matrices designated $\mathbf{C}_g$, $\mathbf{G}_m$, and $\mathbf{G}_d$, respectively.

We begin by finding $\mathbf{Y}_{1,1}$. When port 2 is shorted at all harmonics, $c_g(t)$ and C_f are in parallel, so we can immediately write

$$\mathbf{I}_1 = \{[j\Omega(\mathbf{C}_g + \mathbf{C}_f\mathbf{1})]^{-1} + \mathbf{R}_g\mathbf{1}\}^{-1}\mathbf{V}_1 \tag{3.136}$$

and $\mathbf{Y}_{1,1}$ is found by comparing (3.136) to (3.135). $\mathbf{Y}_{2,1}$ is just a little more trouble. When the output is shorted,

$$\mathbf{V}_g = [j\Omega(\mathbf{C}_g + \mathbf{C}_f\mathbf{1})]^{-1}\mathbf{I}_1 \tag{3.137}$$

and

$$\mathbf{I}_2 = (\mathbf{G}_m - j\Omega\mathbf{C}_f\mathbf{1})\mathbf{V}_g \tag{3.138}$$

Substituting (3.136) into (3.137), and the result into (3.138), we finally obtain

$$\begin{aligned}\mathbf{I}_2 = {} & (\mathbf{G}_m - j\Omega\mathbf{C}_f\mathbf{1})[j\Omega(\mathbf{C}_g + \mathbf{C}_f\mathbf{1})]^{-1} \\ & \cdot \{[j\Omega(\mathbf{C}_g + \mathbf{C}_f\mathbf{1})]^{-1} + \mathbf{R}_g\mathbf{1}\}^{-1}\mathbf{V}_1\end{aligned} \tag{3.139}$$

and $\mathbf{Y}_{2,1}$ is found by inspection. $\mathbf{Y}_{2,2}$ and $\mathbf{Y}_{1,2}$ are a little sticky algebraically but straightforward conceptually. From similar manipulations, we obtain

$$\mathbf{I}_2 = \left\{\mathbf{G}_d + \mathbf{Y}_f\left[\mathbf{1} + \mathbf{G}_m\left(\frac{1}{R_g}\mathbf{1} + j\Omega\mathbf{C}_g\right)^{-1}\right]\right\}\mathbf{V}_2 \tag{3.140}$$

and

$$\mathbf{I}_1 = -(\mathbf{1} + jR_g\Omega\mathbf{C}_g)^{-1}\mathbf{Y}_f\mathbf{V}_2 \tag{3.141}$$

from which $\mathbf{Y}_{2,2}$ and $\mathbf{Y}_{1,2}$ are easily identifiable. $\mathbf{Y}_f$ is defined as

$$\mathbf{I}_f = \mathbf{Y}_f\mathbf{V}_2 \tag{3.142}$$

where $\mathbf{I}_f$ is the current in C_f. $\mathbf{Y}_f$ is

$$\mathbf{Y}_f = \left[\left(\frac{1}{R_g}\mathbf{1} + j\Omega\mathbf{C}_g\right)^{-1} - \frac{j}{C_f}\Omega^{-1}\right]^{-1} \tag{3.143}$$

3.4.2.3 Example: Two-Port Formulation

We now calculate the input and output impedances, simultaneous conjugate-match impedances, transducer conversion gain, and maximum available conversion gain of the circuit of the previous example, at a specific pair of input and output frequencies. Figure 3.15 shows the circuit to be analyzed, where the two-port is described by the conversion matrix **Y**, derived in the previous example. The source and load impedances, generally functions of ω, are shown in series with the two-port; shorting either set of terminals loads the input or output port with the appropriate impedance.

We wish to calculate this circuit's gain and impedances at specific input and output frequencies. This means that, with the exception of the input port at the input frequency and the output port at the output frequency, we wish to terminate the ports in their source and load impedances at all mixing frequencies. The source and load impedances at the unwanted mixing frequencies are then absorbed into the network, and we are left with a conventional two-port, describable by a simple 2 × 2 Y matrix. The only feature that would distinguish this matrix from the Y matrix of a time-invariant network is that it represents input and output phasors at different frequencies, and if one of those frequencies is a lower sideband, its voltage and current are conjugate quantities.

We begin by putting the source and load impedances into a compatible two-port conversion matrix representation. This is

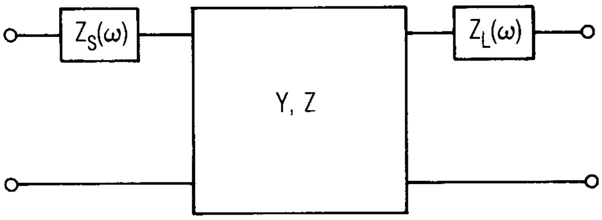

Figure 3.15 Circuit of the example. The block *Y*, *Z* is the circuit in Figure 3.14.

$$\mathbf{Z}_t = \begin{bmatrix} \mathbf{Z}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_L \end{bmatrix} \tag{3.144}$$

where $\mathbf{Z}_s$ and $\mathbf{Z}_L$ are diagonal matrices of the form shown in (3.128). Following the notation for mixing products (3.110), we let ω_q be the input frequency and ω_r be the output frequency. The source and load impedances at these frequencies, $Z_s(\omega_q)$ and $Z_L(\omega_r)$, respectively, are set to zero in (3.144), because we want them to remain external to the circuit; the impedances at other frequencies are retained and are absorbed into the circuit. Following the rule for conventional two-ports, the impedance-form conversion matrix of the terminated network, $\mathbf{Z}_a$, is

$$\mathbf{Z}_a = \mathbf{Z}_t + \mathbf{Y}^{-1} \tag{3.145}$$

The admittance-form matrix for the combination of the FET and the source and load impedances is

$$\mathbf{Y}_a = \mathbf{Z}_a^{-1} \tag{3.146}$$

At this point, $\mathbf{Y}_a$ still relates two voltage vectors to two current vectors and has the form

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{a;1,1} & \mathbf{Y}_{a;1,2} \\ \mathbf{Y}_{a;2,1} & \mathbf{Y}_{a;2,2} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \tag{3.147}$$

We now reduce (3.147) to a simple 2×2 admittance matrix by terminating the ports at all unwanted mixing frequencies. To terminate the output port at all frequencies other than ω_r we set $\mathbf{V}_2$ to zero by shorting the output terminals at those frequencies; similarly, $\mathbf{V}_1$ is zeroed at all frequencies other than ω_q. Setting these voltage components to zero multiplies all the corresponding columns in $\mathbf{Y}_a$ by zero; therefore, those columns can be eliminated. Furthermore, because the input and output are shorted, the current components at those frequencies are not of interest, so the corresponding rows in $\mathbf{Y}_a$ and $\mathbf{I}$ can also be removed. The only terms left in $\mathbf{Y}_a$ can be put into the 2×2 matrix form,

$$\begin{bmatrix} I_1(\omega_q) \\ I_2(\omega_r) \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \begin{bmatrix} V_1(\omega_q) \\ V_2(\omega_r) \end{bmatrix} \tag{3.148}$$

where

$$\begin{aligned} y_{1,1} &= \mathbf{Y}_{a;1,1}(\omega_q, \omega_q) \\ y_{1,2} &= \mathbf{Y}_{a;1,2}(\omega_q, \omega_r) \\ y_{2,1} &= \mathbf{Y}_{a;2,1}(\omega_r, \omega_q) \\ y_{2,2} &= \mathbf{Y}_{a;2,2}(\omega_r, \omega_r) \end{aligned} \tag{3.149}$$

The rest is all downhill. Equation (3.148) can now be used with the usual assortment of Y-matrix relations. For example, if the load admittance is $Y_L(\omega_r)$, the input admittance has the familiar relation,

$$Y_{\text{in}}(\omega_q) = y_{1,1} - \frac{y_{1,2}\, y_{2,1}}{Y_L(\omega_r) + y_{2,2}} \tag{3.150}$$

and with a source admittance $Y_s(\omega_q)$, the output admittance is

$$Y_{\text{out}}(\omega_r) = y_{2,2} - \frac{y_{2,1}\, y_{1,2}}{Y_s(\omega_q) + y_{1,1}} \tag{3.151}$$

Note that if $r < 0$, the output admittance is conjugate; if $q < 0$, the input admittance is conjugate. In these cases, the conjugate of the load or source admittance must also be used in (3.150) and (3.151), respectively. The equation for transducer conversion gain, in terms of Y parameters, is

$$G_t = \frac{4\operatorname{Re}\{Y_s(\omega_q)\}\operatorname{Re}\{Y_L(\omega_r)\}|y_{2,1}|^2}{|[y_{1,1} + Y_s(\omega_q)][y_{2,2} + Y_L(\omega_r)] - y_{1,2}\, y_{2,1}|^2} \tag{3.152}$$

The Linvill stability factor, c, is

$$c = \frac{|y_{1,2}\, y_{2,1}|}{2\operatorname{Re}\{y_{1,1}\}\operatorname{Re}\{y_{2,2}\} - \operatorname{Re}\{y_{2,1}\, y_{1,2}\}} \tag{3.153}$$

If $c < 1$, the circuit is unconditionally stable, and no passive source impedance at ω_q or load at ω_r can cause oscillation. If $c < 1$, the maximum available conversion gain (MAG) and simultaneous conjugate match impedances $Y_{s,\,\mathrm{opt}}(\omega_q)$, $Y_{L,\,\mathrm{opt}}(\omega_r)$ are defined. They are

$$MAG = \frac{|y_{2,\,1}|^2}{2\mathrm{Re}\{y_{1,\,1}\}\mathrm{Re}\{y_{2,\,2}\} - \mathrm{Re}\{y_{2,\,1}y_{1,\,2}\} + T_y} \tag{3.154}$$

and

$$\mathrm{Im}\{Y_{s,\,\mathrm{opt}}(\omega_q)\} = -\mathrm{Im}\{y_{1,\,1}\} + \frac{\mathrm{Im}\{y_{2,\,1}y_{1,2}\}}{2\mathrm{Re}\{y_{2,\,2}\}} \tag{3.155}$$

$$\mathrm{Re}\{Y_{s,\,\mathrm{opt}}(\omega_q)\} = \frac{T_y}{2\mathrm{Re}\{y_{2,\,2}\}} \tag{3.156}$$

where

$$T_y = [(2\mathrm{Re}\{y_{1,\,1}\}\mathrm{Re}\{y_{2,\,2}\} - \mathrm{Re}\{y_{2,1}y_{1,\,2}\})^2 - |y_{1,\,2}y_{2,\,1}|^2]^{1/2} \tag{3.157}$$

The load impedance $Y_{L,\,\mathrm{opt}}(\omega_r)$ can be found from (3.155) and (3.156) by interchanging $y_{1,1}$ and $y_{2,2}$, and $y_{2,1}$ and $y_{1,2}$.

As is the case in a time-invariant circuit, unconditional stability at the excitation frequency and large-signal excitation level is not adequate to guarantee that the time-varying circuit is stable in a practical sense; for the circuit to be stable in practice, it must be unconditionally stable at all possible input frequencies and large-signal excitation levels. Varying the small-signal excitation frequency for which the Y parameters in (3.148) are determined also varies the higher-order mixing frequencies, and hence the embedding impedances at those frequencies. Stability, therefore, is a function of everything that affects the Y parameters, literally all the characteristics of the circuit and its large-signal excitation.

It is important to recognize that small-signal and large-signal stability are interrelated. To explain why this is so, we must note that a fundamental assumption in the conversion matrix theory is that small-signal voltages are small variations (in frequency as well as in magnitude and phase) in the large-signal voltage. The conversion matrix is in fact nothing more than the large-signal Jacobian, a matrix that relates the current and voltage deviations, evaluated at the mixing frequencies instead of the large-signal harmonics. Small-signal oscillation is a process where these variations build up spontaneously and without bound and eventually become indistinguishable from the large-signal voltage. If they occur at a different frequency from the large signal, they may appear as modulation, "snap" phenomena, parasitic oscillation, or other well-known manifestations of instability in nonlinear circuits.

The two-port conversion matrix of (3.148) is in admittance form only because an admittance-form conversion matrix is usually most convenient. It need not be expressed in this form, however; in fact, it can be converted to any two-port matrix form desired, such as an S matrix or even a T matrix (transfer-scattering matrix). The procedure for converting the Y matrix to one of these forms is precisely the same as for any other scalar matrix. For example, the S matrix is found from the Y matrix as

$$\mathbf{S} = (\mathbf{1} + \mathbf{Y}_{\text{norm}})^{-1}(\mathbf{1} + \mathbf{Y}_{\text{norm}}) \tag{3.158}$$

where $\mathbf{Y}_{\text{norm}}$ is the Y matrix (3.148) normalized to the S parameters' reference admittance. The interpretation of lower-sideband quantities (q, $r < 0$) in the S matrix may be a little confusing. For example, if $q = -1$ and $r = 0$, a common situation, the S matrix has the form

$$\begin{bmatrix} b_1^*(\omega_{-1}) \\ b_2(\omega_0) \end{bmatrix} = \begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{bmatrix} \begin{bmatrix} a_1^*(\omega_{-1}) \\ a_2(\omega_0) \end{bmatrix} \tag{3.159}$$

where $s_{1,1}$ is the conjugate of the input reflection coefficient:

$$\Gamma_{\text{in}}^* = s_{1,1} = \left. \frac{b_1^*(\omega_{-1})}{a_1^*(\omega_{-1})} \right|_{a_2(\omega_0) = 0} \tag{3.160}$$

and $|s_{2,1}|^2$ is, as usual, the transducer gain

$$G_t = |s_{2,1}|^2 = \left. \left| \frac{b_2(\omega_0)}{a_1^*(\omega_{-1})} \right|^2 \right|_{a_2(\omega_0) = 0} \tag{3.161}$$

The fact that a_1 is conjugate in (3.161) does not change the magnitude of $s_{2,1}$. Fortunately, the fact that the definitions of $s_{2,1}$ and $s_{1,2}$ include one conjugate and one nonconjugate quantity rarely is a problem; the properties that are usually of most interest—gain, impedances, and stability—are scalar.

When the conversion-matrix formulation is used in this manner, it has significant advantages over multitone harmonic-balance analysis. Such characteristics as simultaneous conjugate match impedances and maximum available gain can be calculated easily; these would be much more difficult to determine with harmonic-balance analysis. Even calculating a set of two-port S parameters would require two harmonic-balance analyses. When conversion-matrix analysis is used, S parameters can be calculated with only one single-tone harmonic-balance analysis; the subsequent conversion-matrix manipulations are computationally inexpensive, especially compared to two-tone harmonic balance. A disadvantage is the lack of nonlinear calculations, but these can be included as well, as we shall see in Section 3.5.

3.4.3 Nodal Formulation

In order to use conversion matrices in a general-purpose circuit analysis program, we need a general-purpose method for formulating the equations. In static linear analysis, we often formulate the equations as an indefinite admittance matrix, which we then reduce to a conventional, nodal admittance matrix. We can do the same thing with conversion matrices. We end up with a set of equations that looks like (3.134), and we use manipulations identical to those of Section 3.4.2.3 to obtain S parameters, port reflection coefficients, gain, or other characteristics of interest.

Consider a time-varying admittance element, whose conversion matrix is $\mathbf{Y}_c$, connected between nodes i and j, as shown in Figure 3.16. Let $\mathbf{I}_i$ and $\mathbf{I}_j$ be the vectors of current in the element connected between nodes i and j, respectively, and $\mathbf{V}_i$ and $\mathbf{V}_j$ be the voltages. These voltages and currents have the form of the voltage and current vectors in (3.116) and (3.117). The current in the branch is

$$\mathbf{I}_i = \mathbf{Y}_c(\mathbf{V}_i - \mathbf{V}_j) \tag{3.162}$$

and

$$\mathbf{I}_j = \mathbf{Y}_c(\mathbf{V}_j - \mathbf{V}_i) \tag{3.163}$$