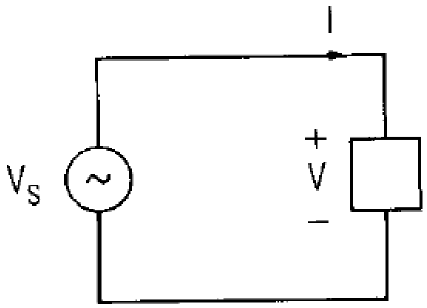
NONLINEAR PHENOMENA

- The examination of new frequencies generated in nonlinear circuits does not tell the whole story of nonlinear effects, especially the effects of nonlinearities on RF systems.
- Many types of nonlinear phenomena have been defined; the power series techniques can show how these arise from the nonlinearities in individual components or circuit elements.
- The phenomena described hereinafter are often considered to be entirely different; we shall see, however, that they are simply manifestations of the same nonlinearities.

NONLINEAR PHENOMENA

- Given the two-terminal nonlinear resistor excited directly by a voltage source.



$$I = aV + bV^2 + cV^3$$

- *Vs* is a two-tone excitation of the form $V_s = v_s(t) = V_1\cos(\omega_1 t) + V_2\cos(\omega_2 t)$

- The three current components are

$$i_a(t) = av_s(t) = aV_1\cos(\omega_1 t) + aV_2\cos(\omega_2 t)$$

$$i_b(t) = bv_s^2(t) = \frac{b}{2}\{V_1^2 + V_2^2 + V_1^2\cos(2\omega_1 t) + V_2^2\cos(2\omega_2 t) + 2V_1V_2[\cos((\omega_1+\omega_2)t) + \cos((\omega_1-\omega_2)t)]\}$$

$$\begin{aligned} i_c(t) = cv_s^3(t) = \frac{c}{4}\{&V_1^3\cos(3\omega_1 t) + V_2^3\cos(3\omega_1 t) \\ &+ 3V_1^2V_2[\cos((2\omega_1+\omega_2)t) + \cos((2\omega_1-\omega_2)t)] \\ &+ 3V_1V_2^2[\cos((\omega_1+2\omega_2)t) + \cos((\omega_1-2\omega_2)t)] \\ &+ 3(V_1^3 + 2V_1V_2^2)\cos(\omega_1 t) \\ &+ 3(V_2^3 + 2V_1^2V_2)\cos(\omega_2 t)\} \end{aligned}$$

NONLINEAR PHENOMENA

- A closer examination of the generated frequencies shows that all occur at a linear combination of the two excitation frequencies; that is, at the frequencies

$$\omega_{m,n} = m\omega_1 + n\omega_2$$

- where m, n = ..., −3, −2, −1, 0, 1, 2, 3, ... The term ω_m, n is called a *mixing frequency*

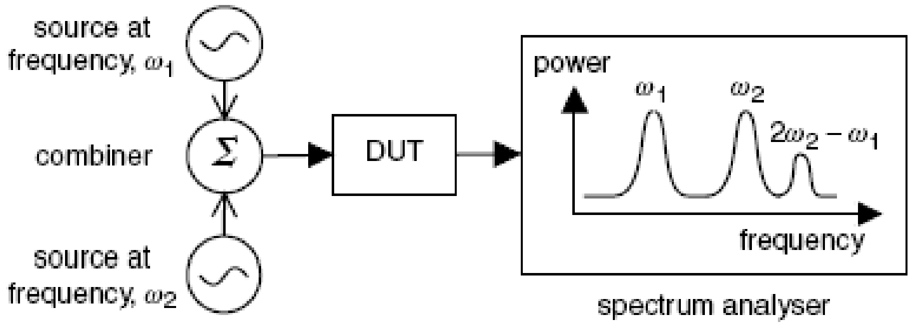
- **Harmonic Generation**

- the generation of harmonics of the excitation frequency or frequencies, are at $m\omega_1$, $m\omega_2$.
- In narrow-band systems, harmonics are not a serious problem because they are far removed in frequency from the signals of interest and inevitably are rejected by filters.
- In others, such as transmitters, harmonics may interfere with other communications systems and must be reduced by filters or other means.

NONLINEAR PHENOMENA

Intermodulation Distortion

- All the mixing frequencies that arise from linear combinations of two or more tones are often called *intermodulation* (IM) *products*.
- IM products generated in an amplifier or communications receiver often present a serious problem, because they represent spurious signals that interfere with, and can be mistaken for, desired signals.
- IM products are generally much weaker than the signals that generate
- two or more very strong signals, which may be outside the receiver's passband, generate an IM product that is within the receiver's passband and obscures a weak, desired signal.
- The IM products of greatest concern are usually the third-order ones that occur at $2\omega_1 - \omega_2$ and $2\omega_2 - \omega_1$, because they are the strongest



NONLINEAR PHENOMENA

Saturation and Desensitization

- In order to describe saturation, we consider one-tone signal V1 at ω_1, the current at fundamental *i*1(*t*) is:

$$i_1(t) = \left(aV_1 + \frac{3}{4}cV_1^3\right)\cos(\omega_1 t)$$

- If the coefficient *c* of the cubic term is negative, the response current saturates; that is, it does not increase at a rate proportional to the increase in excitation voltage.
- Saturation occurs in all circuits because the available output power is finite. If a circuit such as an amplifier is excited by a large and a small signal, and the large signal drives the circuit into saturation, gain is decreased for the weak signal as well. Saturation therefore causes a decrease in system sensitivity, called *desensitization*.

NONLINEAR PHENOMENA

Cross Modulation

- Cross modulation is the transfer of modulation from one signal to another in a nonlinear circuit. Consider the excitation:

$$V_s = v_s(t) = V_1 \cos(\omega_1 t) + (1 + m(t))\cos(\omega_2 t)$$

- where *m*(*t*) is a modulating waveform; $|m(t)| < 1$. Substituting the signal in the nonlinearity gives an expression for the third-degree term, where the frequency component in *ic*(*t*) at ω1:

$$i_c'(t) = \frac{3}{2} c V_1 V_2^2 (1 + 2m(t) + m^2(t))\cos(\omega_1 t)$$

- where a distorted version of the modulation of the ω2 signal has been transferred to the ω1 carrier.
- This transfer occurs simply because the two signals are simultaneously present in the same circuit. The effect depends upon the magnitude of the coefficient *c* and the strength of the interfering signal ω2.

NONLINEAR PHENOMENA

AM-to-PM Conversion

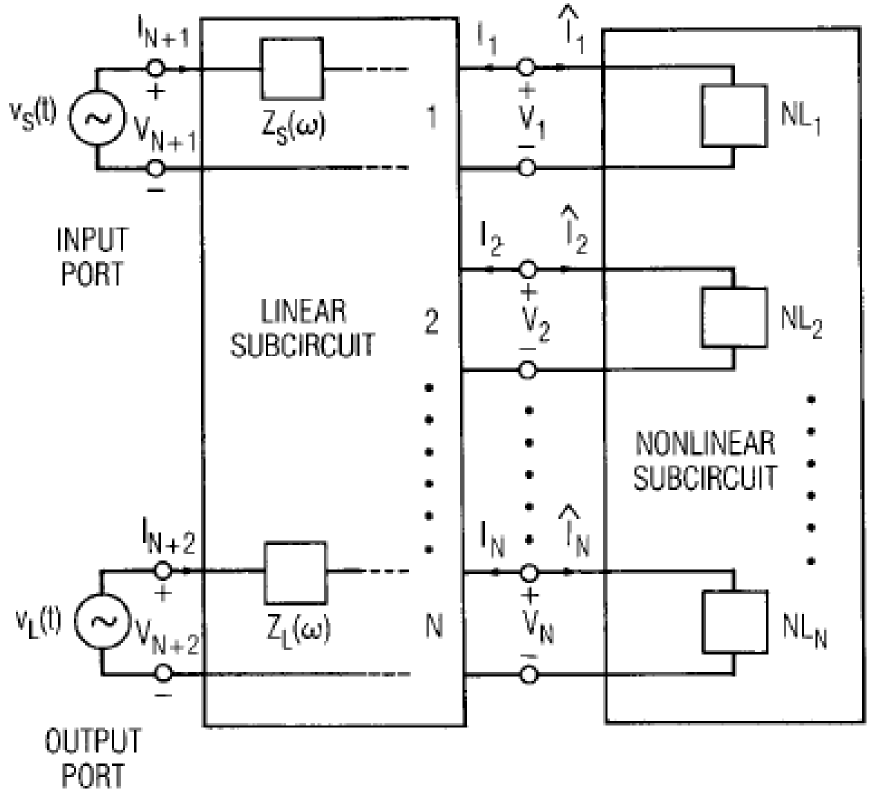
- AM-to-PM conversion is a phenomenon wherein changes in the amplitude of a signal applied to a nonlinear circuit cause a phase shift.
- This form of distortion can have serious consequences if it occurs in a system in which the signal's phase is important; for example, phase- or frequency modulated communication systems.
- This possibility is not predicted by the power series because these equations describe a memoryless nonlinearity. In a circuit having reactive nonlinearities, however, it is possible for a phase difference to exist.
- The response current at $\omega 1$ in the nonlinear circuit element is of the form:

$$I_1(\omega_1) = aV_1 + \frac{3}{4}cV_1^3\exp(j\theta)$$

- where θ is the phase difference. Even if θ remains constant with amplitude, the phase of *I*1 changes with variations in *V*1.
- AM-to-PM conversion is most serious as the circuit is driven into saturation.

HARMONIC-BALANCE ANALYSIS

- In general, RF circuits have a large number of both linear and nonlinear circuit elements.
- These can be grouped as shown to form two subcircuits, one linear and the other nonlinear.
- The linear subcircuit can be treated as a multiport and described by its Y parameters, S parameters, or by some other multiport matrix.
- The nonlinear elements are modeled by their global *I*/*V* or *Q*/*V* characteristics, and must be analyzed in the time domain.
- Thus, the circuit is reduced to an (N + 2)-port network, with nonlinear elements connected to N of the ports and voltage sources connected to the other two ports.



HARMONIC-BALANCE ANALYSIS

- The voltages and currents at each port can be expressed in the time or the frequency domain; because of the nonlinear elements, however, the port voltages and currents have frequency components at harmonics of the excitation.
- Although in theory an infinite number of harmonics exist at each port, we shall assume throughout this chapter that the dc component and the first K harmonics (i.e., $k = 0 \ldots K$) describe all the voltages and currents adequately.
- The circuit is successfully analyzed when either the steady-state voltage or current waveforms at each port are known.
- Alternatively, knowledge of the frequency components at all ports constitutes a solution, because the frequency components and time waveforms are related by the Fourier series.
- If, for example, we know the frequency domain port voltages, we can use the Y-parameter matrix of the linear subcircuit to find the port currents.
- The port currents can also be found by inverse-Fourier transforming the voltages to obtain their time-domain waveforms and calculating the current waveforms from the nonlinear elements' *I*/*V* equations.

HARMONIC-BALANCE ANALYSIS

- The idea of harmonic balance is to find a set of port voltage waveforms (or, alternatively, the harmonic voltage components) that give the same currents in both the linear-network equations and the nonlinear-network equations; that is, the currents satisfy Kirchoff's current law.
- If we express the frequency components of the port currents as vectors, Kirchoff's current law requires that

$$\begin{bmatrix} I_{1,0} \\ I_{1,1} \\ I_{1,2} \\ \cdots \\ I_{1,K} \\ I_{2,0} \\ I_{2,1} \\ \cdots \\ I_{2,K} \\ \cdots \\ I_{N,K} \end{bmatrix} + \begin{bmatrix} \hat{I}_{1,0} \\ \hat{I}_{1,1} \\ \hat{I}_{1,2} \\ \cdots \\ \hat{I}_{1,K} \\ \hat{I}_{2,0} \\ \hat{I}_{2,1} \\ \cdots \\ \hat{I}_{2,K} \\ \cdots \\ \hat{I}_{N,K} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

HARMONIC-BALANCE ANALYSIS

- First we consider the linear subcircuit
- The admittance equations are described in the matrix form by:

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \\ \cdots \\ \mathbf{I}_N \\ \mathbf{I}_{N+1} \\ \mathbf{I}_{N+2} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{1,1} & \mathbf{Y}_{1,2} & \cdots & \mathbf{Y}_{1,N} & \mathbf{Y}_{1,N+1} & \mathbf{Y}_{1,N+2} \\ \mathbf{Y}_{2,1} & \mathbf{Y}_{2,2} & \cdots & \mathbf{Y}_{2,N} & \mathbf{Y}_{2,N+1} & \mathbf{Y}_{2,N+2} \\ \mathbf{Y}_{3,1} & \mathbf{Y}_{3,2} & \cdots & \mathbf{Y}_{3,N} & \mathbf{Y}_{3,N+1} & \mathbf{Y}_{3,N+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{Y}_{N,1} & \mathbf{Y}_{N,2} & \cdots & \mathbf{Y}_{N,N} & \mathbf{Y}_{N,N+1} & \mathbf{Y}_{N,N+2} \\ \mathbf{Y}_{N+1,1} & \mathbf{Y}_{N+1,2} & \cdots & \mathbf{Y}_{N+1,N} & \mathbf{Y}_{N+1,N+1} & \mathbf{Y}_{N+1,N+2} \\ \mathbf{Y}_{N+2,1} & \mathbf{Y}_{N+2,2} & \cdots & \mathbf{Y}_{N+2,N} & \mathbf{Y}_{N+2,N+1} & \mathbf{Y}_{N+2,N+2} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \cdots \\ \cdots \\ \mathbf{V}_N \\ \mathbf{V}_{N+1} \\ \mathbf{V}_{N+2} \end{bmatrix}$$

$$\mathbf{I}_n = \begin{bmatrix} I_{n,0} \\ I_{n,1} \\ \cdots \\ I_{n,K} \end{bmatrix} \qquad \mathbf{V}_n = \begin{bmatrix} V_{n,0} \\ V_{n,1} \\ \cdots \\ V_{n,K} \end{bmatrix} \qquad Y_{m,n} = diag[Y_{m,n}(k\omega_p)] \qquad k = 0, 1, 2, \ldots, K$$

- The e … the … matrix **Y***m,n* are all submatrices; each submatrix is a diagonal, whose elements are the values *Ym,n* at each harmonic of the fundamental excitation frequency, $k\omega_p$, $k = 0 \ldots K$:

HARMONIC-BALANCE ANALYSIS

- The elements of the admittance matrix **Y***m,n* are all submatrices; each submatrix is a diagonal, whose elements are the values *Ym,n* at each harmonic of the fundamental excitation frequency, $k\omega_p$, $k = 0 \ldots K$:

$$Y_{m,n} = diag[Y_{m,n}(k\omega_p)] \qquad k = 0, 1, 2, \ldots, K$$

- Considering the $\mathbf{V}_{N+1}$ and $\mathbf{V}_{N+2}$, the excitation vectors

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \ldots \\ \mathbf{I}_N \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{1,N+1} & \mathbf{Y}_{1,N+2} \\ \mathbf{Y}_{2,N+1} & \mathbf{Y}_{2,N+2} \\ \ldots & \ldots \\ \mathbf{Y}_{N,N+1} & \mathbf{Y}_{N,N+2} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{N+1} \\ \mathbf{V}_{N+2} \end{bmatrix} + \begin{bmatrix} \mathbf{Y}_{1,1} & \mathbf{Y}_{1,2} & \ldots & \mathbf{Y}_{1,N} \\ \mathbf{Y}_{2,1} & \mathbf{Y}_{2,2} & \ldots & \mathbf{Y}_{2,N} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{Y}_{N,1} & \mathbf{Y}_{N,2} & \ldots & \mathbf{Y}_{N,N} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \ldots \\ \mathbf{V}_N \end{bmatrix}$$

$$\mathbf{I} = \mathbf{I}_s + \mathbf{Y}_{N \times N}\mathbf{V}$$

- This transformation allows us to express theharmonic-balance equations as functions of currents at only the firstthrough *N*th ports, the ones connected to nonlinear elements

HARMONIC-BALANCE ANALYSIS

- **The Nonlinear Subcircuit**
- we assume that the nonlinear elements are all voltage controlled (these assumptions do not limit us severely)
- Inverse Fourier transforming the voltages at each port gives the time-domain voltage waveforms at each port:

$$\mathcal{F}^{-1}\{\mathbf{V}_n\} \rightarrow v_n(t)$$

- We first examine nonlinear capacitors.
- Because the port voltages uniquely determine all voltages in the network, a capacitor's charge waveform can be expressed as a function of those voltages:

$$q_n(t) = f_{qn}(v_1(t), v_2(t), \ldots, v_N(t))$$

- Fourier transforming the charge waveform at each port gives the charge vectors for the capacitors at each port:

$$\mathcal{F}\{q_n(t)\} \rightarrow \mathbf{Q}_n$$

Q charge vector

HARMONIC-BALANCE ANALYSIS

- The nonlinear-capacitor current is the time derivative of the charge waveform. Taking the time derivative corresponds to multiplying by $j\omega$ in the frequency domain, so

$$i_{c,n}(t) = \frac{dq_n(t)}{dt} \leftrightarrow jk\omega_p Q_{n,k} \qquad k = 0, 1, \dots, K \qquad \mathbf{I}_c = j\Omega\mathbf{Q}$$

- where Ω is the diagonal matrix
- Similarly, the current in a nonlinear conductance or a controlled current source is

$$i_{g,n}(t) = f_n(v_1(t), v_2(t), \dots, v_N(t))$$

- Fourier transforming these gives

$$\mathcal{F}\{i_{g,n}(t)\} \rightarrow \mathbf{I}_{G,n} \qquad \mathbf{I}_G = \begin{bmatrix} \mathbf{I}_{G,1} \\ \mathbf{I}_{G,2} \\ \dots \\ \mathbf{I}_{G,N} \end{bmatrix}$$

HARMONIC-BALANCE ANALYSIS

- Substituting in the KVC gives the expression

$$\mathbf{F}(\mathbf{V}) = \mathbf{I}_s + \mathbf{Y}_{N \times N}\mathbf{V} + j\Omega\mathbf{Q} + \mathbf{I}_G = \mathbf{0}$$

- This equation represents a test to determine whether a trial set of port voltage components is the correct one; that is, if **F**(**V**) = **0**, then **V** is a valid solution.

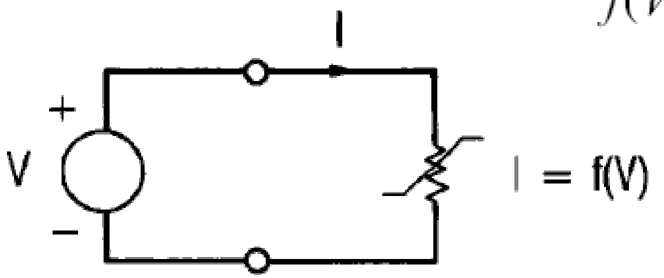
LARGE-SIGNAL/SMALL-SIGNAL ANALYSIS

- *Large-signal/small-signal analysis*, or *conversion matrix analysis*, is useful for a large class of problems wherein a nonlinear device is driven, or “pumped,” by a single large sinusoidal signal; another signal, much smaller, is applied; and we seek only the linear response to the small signal.
- The most common application of this technique is in the design of mixers and in nonlinear noise analysis.
- It cannot be used for determining saturation or intermodulation distortion in mixers, but it is a good method for calculating a mixer’s conversion efficiency and its RF and IF port impedances.
- The results of the harmonic-balance analysis can be used for finding LO voltage and current waveforms, and LO port impedance.

LARGE-SIGNAL/SMALL-SIGNAL ANALYSIS

Conversion Matrix Formulation

- Let’s consider a nonlinear resistive element driven by a large-signal voltage, *V*, generating a current *I*.
- we can find the incremental small-signal current by assuming that *V* consists of the sum of a large-signal component *V*0 and a small-signal component *v*. The current resulting from this excitation can be found by expanding *f* (*V*0 + *v*) in a Taylor series,



$$f(V_0 + v) = f(V_0) + \frac{d}{dV} f(V)\bigg|_{V = V_0} v + \frac{1}{2}\frac{d^2}{dV^2} f(V)\bigg|_{V = V_0} v^2 + \frac{1}{6}\frac{d^3}{dV^3} f(V)\bigg|_{V = V_0} v^3 + \ldots$$

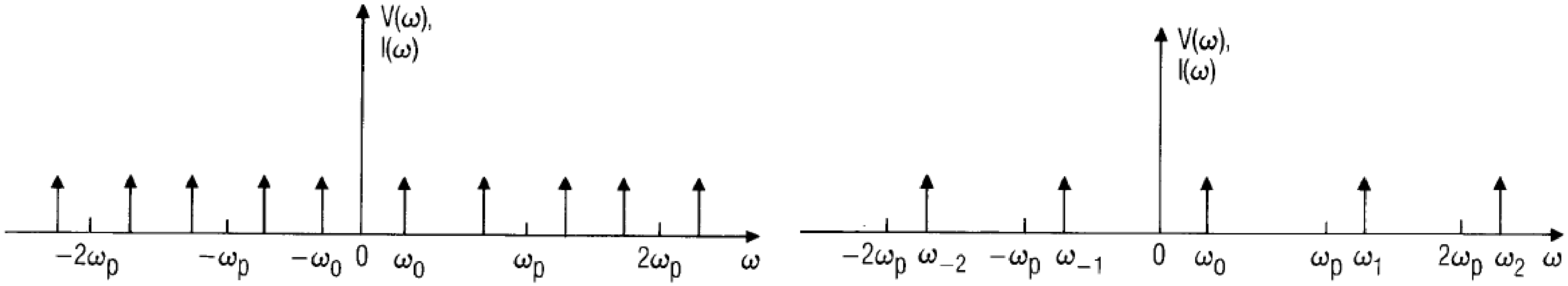
- The small-signal, incremental current is found by subtracting the large-signal component of the current,

$$i(v) = I(V_0 + v) - I(V_0)$$

- If *v << V0, v2, v3, ... are negligible*

$$i(t) = \frac{d}{dV} f(V)\bigg|_{V = V_L(t)} v(t) \quad \longrightarrow \quad i(t) = g(t)v(t)$$

- *The mixing frequencies are:* $\omega_n = \omega_0 + n\omega_p$ $\qquad \omega_0 = |\omega_1 - \omega_p|$



- The frequency-domain currents and voltages in a time-varying circuit element are related by a *conversion matrix, t*hat represents a time-varying conductance.
- The small-signal voltage and current can be expressed as

$$i'(t) = \sum_{n=-\infty}^{\infty} I_n \exp(j\omega_n t) \qquad v'(t) = \sum_{n=-\infty}^{\infty} V_n \exp(j\omega_n t)$$

- The conductance waveform *g(t) can* be expressed by its Fourier series:

$$g(t) = \sum_{n=-\infty}^{\infty} G_n \exp(jn\omega_p t)$$

- voltage and current are related by Ohm's law

$$i'(t) = g(t)v'(t) \longrightarrow \sum_{k=-\infty}^{\infty} I_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_n V_m \exp(j\omega_{m+n} t)$$

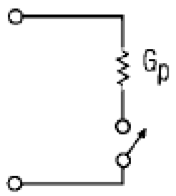
- Equating terms on both sides results in a set of equations that can be expressed in matrix form:

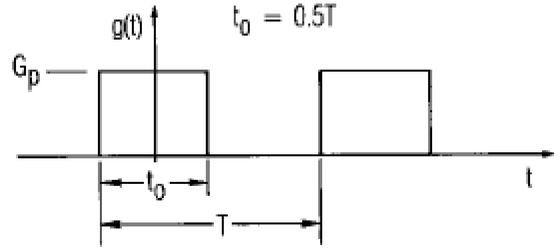
- limit of $n = N$ *for* I_n *and* V_n*, and* $n = 2N$ *for* G_n*,* assuming that V_n*,* I_n*, and* G_n *are negligible beyond these limits.*
- the negative-frequency components (V_n, I_n *where* $n < 0$*) are* shown as conjugate. By definition ω_n *is negative when* $n < 0$*;* positive- and negative-frequency components are related as $V_{-n} = V_n^*$ and $I_{-n} = I_n^*$.
- Thus the conversion matrix relates ordinary phasor voltages to currents at each mixing frequency.
- the conversion matrix is completely compatible with conventional linear, sinusoidal steady-state analysis.

$$
\begin{bmatrix} I_{-N}^* \\ I_{-N+1}^* \\ I_{-N+2}^* \\ \cdots \\ \cdots \\ I_{-1}^* \\ I_0 \\ I_1 \\ \cdots \\ \cdots \\ I_N \end{bmatrix}
=
\begin{bmatrix}
G_0 & G_{-1} & G_{-2} & \cdots & G_{-2N} \\
G_1 & G_0 & G_{-1} & \cdots & G_{-2N+1} \\
G_2 & G_1 & G_0 & \cdots & G_{-2N+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
G_{N-1} & G_{N-2} & G_{N-3} & \cdots & G_{-N-1} \\
G_N & G_{N-1} & G_{N-2} & \cdots & G_{-N} \\
G_{N+1} & G_N & G_{N-1} & \cdots & G_{-N+1} \\
\cdots & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \\
G_{2N} & G_{2N-1} & G_{2N-2} & \cdots & G_0
\end{bmatrix}
\begin{bmatrix} V_{-N}^* \\ V_{-N+1}^* \\ V_{-N+2}^* \\ \cdots \\ \cdots \\ V_{-1}^* \\ V_0 \\ V_1 \\ \cdots \\ \cdots \\ V_N \end{bmatrix}
$$

Example: diode in switching conditions

- It consists of a conductance in series with a switch; the switch is opened and closed with a duty cycle of 0.5, so the combination has the waveform





- Its Fourier series, when *t0 = 0.5T, is*

$$g(t) = G_p[0.5 + 0.318\exp(j\omega_p t) + 0.318\exp(-j\omega_p t) - 0.106\exp(j3\omega_p t) - 0.106\exp(-j3\omega_p t) + 0.064\exp(j5\omega_p t) + 0.064\exp(-j5\omega_p t) + \dots$$

- The conversion matrix when 2*N = 6 is*

$$\mathbf{G} = G_p \begin{bmatrix} 0.5 & 0.318 & 0 & -0.106 & 0 & 0.064 & 0 \\ 0.318 & 0.5 & 0.318 & 0 & -0.106 & 0 & 0.064 \\ 0 & 0.318 & 0.5 & 0.318 & 0 & -0.106 & 0 \\ -0.106 & 0 & 0.318 & 0.5 & 0.318 & 0 & -0.106 \\ 0 & -0.106 & 0 & 0.318 & 0.5 & 0.318 & 0 \\ 0.064 & 0 & -0.106 & 0 & 0.318 & 0.5 & 0.318 \\ 0 & 0.064 & 0 & -0.106 & 0 & 0.318 & 0.5 \end{bmatrix}$$